
# Game Theory with(out) Tears

**Vito Fragnelli**
University of Eastern Piedmont
*vito.fragnelli@mfn.unipmn.it*

Centro di Cultura Scientifica "Alessandro Volta"
Universitas Studiorum Insubriae
UPO Università del Piemonte Orientale

# 1 Theory of Games

## 1.1 Introduction

Game Theory deals with situations whose final result depends on the choices of several decision-makers, the *players*; their targets may be

- common, even if not identical
- different
- opposite

Random elements are allowed

The name is after the book *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944)

**Example 1.1 (Prisoner's Dilemma)**

| $I/II$ | $C$ | $NC$ |
|---|---|---|
| $C$ | $-5, -5$ | $-1, -6$ |
| $NC$ | $-6, -1$ | $-2, -2$ |

◇

Harsanyi Classification (1966):

| | |
|---|---|
| Non-cooperative games | Binding agreements are not allowed |
| Cooperative games | Binding agreements are allowed |

- It is possible to assume that in a non-cooperative setting the players cannot communicate, because communications may influence their choices
- Cooperative games may be divided in two subclasses: non transferable utility games (NTU-Games) or without side payments and transferable utility games (TU-Games) or with side payments, which are a special case of NTU-Games

### 1.2 Representation of a Game

- Extensive form - von Neumann (1928) and Kuhn (1953)
- Strategic form - Shubik (1982); normal form - von Neumann and Morgenstern (1944)
- Characteristic form - von Neumann and Morgenstern (1944); for cooperative games only

**Definition 1.1**

- *A payoff function assigns a value to each player for each possible termination of the game*
- *A strategy is a function that assigns a move to a player for each possible situation of the game in which he is the decision-maker*

The numbers used in the payoff function depend on the concepts of *preference* and of *utility á la von Neumann-Morgenstern*
A strategy is an “action plan” of a player that allows him selecting a “move” for each situation of a game

### 1.3 Extensive Form

Description of the game, specifying:

- the situations of the game
- the player that has to make a decision
- the possible moves, with the corresponding probability distribution, if any
- the information sets
- the payoffs of the players for each termination of the game

It is a very rich and detailed representation but it may result difficult to manage

A simpler approach is to use a *tree representation*, or *decisional tree*:

| | |
|---|---|
| nodes | situations of the game and which player is involved |
| outgoing arcs | set of moves of the player |
| terminal nodes | the payoffs of the players are associated to these nodes |

Before introducing the payoffs, the representation corresponds to the *game form*

### 1.4 Strategic Form

$2n$-uple $(\Sigma_1, \Sigma_2, \ldots, \Sigma_n, f_1, f_2, \ldots, f_n)$ where:

$\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ non-empty sets of the strategies of the players

$f_1, f_2, \ldots, f_n$ real valued function $f_i : \prod_{k=1,\ldots,n} \Sigma_k \rightarrow \mathbb{R}, \qquad i = 1, .., n$

- All the players choose a strategy, i.e. they select a *strategy profile*, consequently $f_i$ assigns a payoff to player $i$
- It is possible to determine the strategic form, given the extensive form, while the reverse task is more complex and not unique
- The elements of the strategic form may be represented via a table, as in Example 1.1
- For a two-player game the table is called *bimatrix*

Referring to the strategic form, the *game form* can be viewed as a triple $\left((\Sigma_i)_{i=1,\ldots,n}, E, h\right)$ where:

$\Sigma_i$ set of strategies of player $i$

$E$ set of final events (or exits)

$h : \prod_{i=1,\ldots,n} \Sigma_i \to E$

It is possible to distinguish between the utility functions:

$$u_i : E \to \mathbb{R}, i = 1, ..., n$$

and the induced utility functions:

$$f_i = u_i \circ h$$
$$f_i : \prod_{i=1,\ldots,n} \Sigma_i \to \mathbb{R}, i = 1, ..., n$$

The result is the strategic form

### 1.5 Characteristic Form

This form is allowed only for cooperative games

## Definition 1.2

- *Given a set of players $N$, each subset $S \subseteq N$ is a* coalition*. $N$ is the* grand coalition
- *The* characteristic function *is a real valued function*

$$v : 2^N \to \mathbb{R} \text{ s.t. } v(\emptyset) = 0$$

*For NTU-Games the characteristic function is denoted by $V$ and it is a set valued function*

$v$ assigns to $S$ the best result that the players in $S$ may obtain *independently* from the choices of the other players

The representation via the characteristic function is called *characteristic form* or *coalitional form*

**Example 1.2 (Simple Majority)**

*Three players have a target; they succeed when at least two players coordinate their efforts*
*This situation may be represented as follows:*

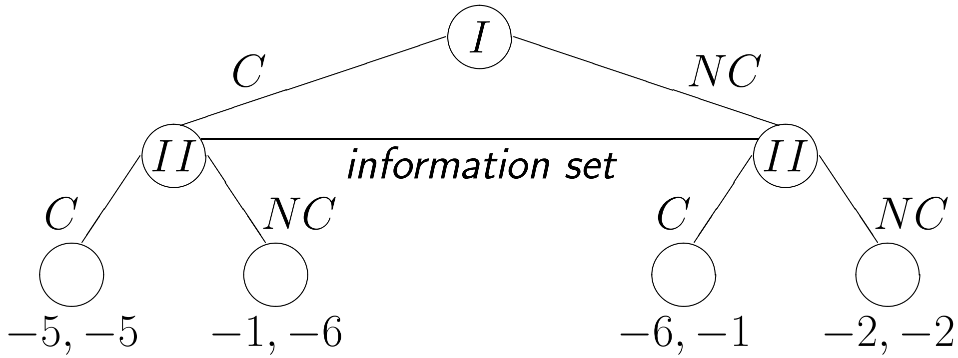
$N = \{1, 2, 3\}$

$v(\emptyset) = v(1) = v(2) = v(3) = 0; v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$ ◇

This description is very "poor", as takes into account only the worth of each coalition ignoring all other aspects: the coalition formation process, the payoff of each agent, etc.

# Example 1.3 (Representation of the Prisoner's Dilemma)

*Extensive form*




*Strategic form*

$$\Sigma_I = \{C, NC\}; \Sigma_{II} = \{C, NC\}$$
$$f_I(C, C) = -5; f_I(C, NC) = -1; f_I(NC, C) = -6; f_I(NC, NC) = -2$$
$$f_{II}(C, C) = -5; f_{II}(C, NC) = -6; f_{II}(NC, C) = -1; f_{II}(NC, NC) = -2$$

*Characteristic form*

$$N = \{I, II\}$$
$$v(\emptyset) = 0; v(I) = v(II) = -5; v(I, II) = -4$$

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### 1.6 Solution Concept

The solution of a game corresponds to a suggestion to the players, possibly all of them, about:

- the strategy to choose if the game is non-cooperative or cooperative without side payments
- how to divide the total payoff of a coalition among its members for cooperative games with side payments

A solution suggests a choice that satisfies global fairness criteria, respecting also the preferences of each player

# 2 Non-cooperative Games

## 2.1 Nash Equilibrium

Players cannot subscribe binding agreements, even if they may be interested in cooperating

The Nash equilibrium (1950) is the simplest and most important solution concept for non-cooperative games
It corresponds to the *global economic equilibrium*

**Definition 2.1** *Given a game $G$, the strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ with $\sigma_i^* \in \Sigma_i$ is an equilibrium, or a Nash equilibrium, if no player has an advantage when he is the unique that changes his strategy:*

$$f_i(\sigma_1^*, \ldots, \sigma_i^*, \ldots, \sigma_n^*) \geq f_i(\sigma_1^*, \ldots, \sigma_i, \ldots, \sigma_n^*), \qquad \forall\ \sigma_i \in \Sigma_i, \forall\ i \in N$$

**Example 2.1 (Prisoner's Dilemma)**

| $I/II$ | $C$ | $NC$ |
|---|---|---|
| $C$ | $-\mathbf{5}, -\mathbf{5}$ | $-1, -6$ |
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- Oddities in the Nash equilibrium:
  - inefficiency
  - non-uniqueness
  - non-existence

# 3 Cooperative Games

## 3.1 TU-Games

Agents may cooperate in order to improve their utility

The cooperation requires:

- the possibility of agreements, i.e. there do not exist antitrust rules
- the possibility of forcing the respect of the agreements, i.e. there exists a *superpartes* authority, accepted by all the agents

Cooperative games are divided in two classes:

- Cooperative games without transferable utility (NTU-Games): the players receive the payoff according to the strategy profile they agreed upon
- Cooperative games with transferable utility (TU-Games): the players may share the total payoff generated by the strategy profile they agreed upon

TU-Games are a special case of NTU-Games

A TU-Game has to satisfy the following three additional hypotheses:

- it is possible to transfer the utility (in a normative sense)
- there exists a common exchange tool, e.g. the money, which allows transferring the utility (in a material sense)
- the utility functions of the players must be equivalent, e.g. they can be linear in the amount of money

In a TU-Game, the decision on the sharing of the total payoff of a coalition is part of the binding agreement

**Definition 3.1** *The* characteristic function *of a $n$-person game is:*

$$v : 2^N \rightarrow \mathbb{R} \text{ with } v(\emptyset) = 0$$

If the payoffs of the players are negative it is convenient to represent the game as a *cost game* $(N, c)$, where $c = -v$

**Example 3.1 (Glove Game)**

*Let $L$ and $R$ be two disjoint sets of players, endowed with some gloves; the players in $L$ own only left gloves, while the players in $R$ own only right gloves. The value of a coalition depends on the number of pairs they are able to form. In general each player is endowed with a single glove. Supposing that $L = \{1, 2\}$, $R = \{3, 4\}$ and the value of each pair is 1, the resulting game is:*

$$\begin{array}{ll} N = \{1,2,3,4\} & \\ v(i) = 0 & \forall\, i \in N \\ v(12) = v(34) = 0 & \\ v(S) = 1 & \text{if } |S| = 2 \text{ and } S \neq \{12\}, S \neq \{34\} \text{ or if } |S| = 3 \\ v(N) = 2 & \end{array}$$

$\diamondsuit$

Generalize the game for any pair of sets $L$ and $R$

**Definition 3.2** *Given a game $G = (N, v)$:*

- *if for every $S \subseteq N$, $v(S) + v(N \setminus S) = v(N)$ for each coalition $S$ then $G$ is a* constant sum game
- *if for every $S \subseteq T \subseteq N$, $v(S) \leq v(T)$ then $G$ is* monotonic
- *if for every pair of disjoint coalitions $S, T \subseteq N$*
  - $v(S \cup T) \geq v(S) + v(T)$ *then $G$ is* superadditive
  - $v(S \cup T) = v(S) + v(T)$ *then $G$ is* additive
  - $v(S \cup T) \leq v(S) + v(T)$ *then $G$ is* subadditive
- *if for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ then $G$ is* convex
- *if for each partition of $N$ $\{S_1, S_2, \ldots, S_k\}$, $\sum_{i=1,\ldots,k} v(S_i) \leq v(N)$ then $G$ is* cohesive

These properties are relevant as many solution concepts consist in dividing the value of the grand coalition among all the players

Check the relations among monotonicity, superadditivity, convexity, cohesiveness

The solutions of a TU Game may be divided in two groups:

- *set solutions*: a set of payoff vectors is associated to the player set
- *point solutions*: a single payoff vector is determined

# 4 Set Solutions for TU Games

## 4.1 Imputations

The payoff of each player may be obtained equally sharing the value of the game among the players, taking in no account the contribution of each player
A different approach is rooted in the analysis of the role of the players

**Definition 4.1** *Given a game* $G = (N, v)$ *an* imputation *is a real vector* $x = (x_1, x_2, \ldots, x_n)$*, where* $x_i$ *represents the amount (payoff) assigned to player* $i \in N$*, such that:*

$$\sum_{i \in N} x_i = v(N) \qquad \text{efficiency}$$

$$x_i \geq v(i) \qquad i \in N \qquad \text{individual rationality}$$

*For a cost game* $G = (N, c)$ *individual rationality is expressed as* $x_i \leq c(i), \forall\ i \in N$

The set of all the imputations is denoted by $E(v)$

**Definition 4.2** *Given a game $G = (N, v)$, if $\sum_{i \in N} v(i) = v(N)$ holds then the unique element of $E(v)$ is $x = (v(1), v(2), \ldots, v(n))$; when $E(v)$ contains more than one vector the game is* essential

An imputation is the first step toward a solution concept that respects the role of the players For an essential game there exist many imputation vectors, so again we have the problem of choosing a solution: given two different imputations $x$ and $y$ there exists at least one player $k$ such that $x_k > y_k$ and at least one player $h$ such that $x_h < y_h$

### 4.2 Core

It is the most interesting set solution for many classes of games
It was introduced by Gillies (1953 and 1959)

$$x(S) \geq v(S) \qquad S \subset N \qquad \text{coalitional rationality}$$

where $x(S) = \sum_{i \in S} x_i$
For a cost game $G = (N, c)$ coalitional rationality is expressed as $x(S) \leq c(S), \forall\ S \subset N$

**Definition 4.3** *Given a game $G = (N, v)$, the core is the set:*

$$C(v) = \{x \in E(v) | x(S) \geq v(S), \forall\ S \subset N\}$$

*A game with non-empty core is called* balanced

- The core may be empty, as for an essential constant sum game
- The core is useful to select which solutions should not be chosen (those not belonging to the core) when the core is non-empty. The emptyness of the core does not imply that the grand coalition does not form, but gives clues on its low stability

Check the relations among superadditivity and balancedness and among cohesiveness and balancedness

## Example 4.1 (Core of Glove Game)

*Referring to the Example 3.1, the core is the set:*

$$C(v) = \{(\alpha, \alpha, 1-\alpha, 1-\alpha) \ s.t. \ 0 \leq \alpha \leq 1\}$$

*In general, let* $L = \{1, ..., n_l\}$ *and* $R = \{1, ..., n_r\}$
*if* $n_l = n_r$*:*

$$C(v) = \{(\alpha, ..., \alpha, 1-\alpha, ..., 1-\alpha) \ s.t. \ 0 \leq \alpha \leq 1\}$$

*if* $n_l < n_r$*:*

$$C(v) = \{(\underbrace{1, ..., 1}_{1,...,n_l}, \underbrace{0, ..., 0}_{1,...,n_r})\}$$

*if* $n_l > n_r$*:*

$$C(v) = \{(\underbrace{0, ..., 0}_{1,...,n_l}, \underbrace{1, ..., 1}_{1,...,n_r})\}$$

- The core of this game represents the situation in which a shortage of complementary goods results in an advantage for the owners of more required goods

# 5 Point Solutions for TU Games

These solutions include the so-called *power indices* and *values*
Power indices are used in simple games in order to evaluate the relevance or power of each player
Values are widely used as allocation rules

## 5.1 Shapley Value

It was introduced by Shapley (1953) and it is rooted in the concept of *marginal contribution*

**Definition 5.1** *Given a game $G = (N, v)$, the Shapley value is the vector $\phi(v)$ whose component $\phi_i$ is the average marginal contribution of player $i$ w.r.t. all the permutations of the players:*

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))]$$

*where $n = |N|$, $\Pi$ is the set of permutations of $N$ and $P(\pi, i)$ is the set of players preceding $i$ in the permutation $\pi$*

Given a TU game, the Shapley value always exists and is unique

If the game is superadditive (subadditive for a cost game) the Shapley value is an imputation as:

$$\sum_{i \in N} \phi_i(v) = v(N)$$
$$\phi_i(v) \geq v(i) \qquad \forall i \in N$$

Prove it

but not always belongs to the core, even if the core is non-empty

If the game is convex (concave, in the case of a cost game) the Shapley lies in the core
Prove it

## Example 5.1 (Assignment Game)

*Consider the game* $v(1) = v(2) = v(3) = v(23) = 0; v(12) = 2; v(13) = v(123) = 5$ *the Shapley value is given by:*

| $Permutations$ | $Marginal\ contributions$ | | |
|---|---|---|---|
| | $Player\ 1$ | $Player\ 2$ | $Player\ 3$ |
| 1 2 3 | $v(1) - v(\emptyset) = 0$ | $v(12) - v(1) = 2$ | $v(123) - v(12) = 3$ |
| 1 3 2 | $v(1) - v(\emptyset) = 0$ | $v(123) - v(13) = 0$ | $v(13) - v(1) = 5$ |
| 2 1 3 | $v(12) - v(2) = 2$ | $v(2) - v(\emptyset) = 0$ | $v(123) - v(12) = 3$ |
| 2 3 1 | $v(123) - v(23) = 5$ | $v(2) - v(\emptyset) = 0$ | $v(23) - v(2) = 0$ |
| 3 1 2 | $v(13) - v(3) = 5$ | $v(123) - v(13) = 0$ | $v(3) - v(\emptyset) = 0$ |
| 3 2 1 | $v(123) - v(23) = 5$ | $v(23) - v(3) = 0$ | $v(3) - v(\emptyset) = 0$ |
| $\phi_i$ | $\frac{17}{6}$ | $\frac{2}{6}$ | $\frac{11}{6}$ |

$\diamondsuit$

Determine the core and check that the Shapley value does not belong to it

Check that the game is not convex

### 5.2 Axiomatization of the Shapley Value

Given a rule $\psi$ that assigns a $n$-dimensional vector in $\mathbb{R}^N$ to each TU-Game $G(N,v)$

1. Symmetry
   If two players $i,j$ are symmetric, i.e. $v(S\cup\{i\}) = v(S\cup\{j\}), \forall\ S \subseteq N\setminus\{i,j\}$ then $\psi_i(v) = \psi_j(v)$

2. Dummy player
   Let $i$ be a dummy player, i.e. $v(S\cup\{i\}) = v(S)+v(i) \quad \forall\ S \subseteq N\setminus\{i\}$ then $\psi_i(v) = v(i)$

3. Additivity or independence (Debatable axiom)
   Given two games with player set $N$ and characteristic functions $v$ and $u$, respectively, let $(u+v)$ the sum game defined as $(u+v)(S) = u(S)+v(S),\ \forall\ S\subseteq N$ then $\psi_i(u+v) = \psi_i(u)+\psi_i(v), \forall\ i\in N$

$\phi$ is the unique efficient vector satisfying the previous axioms

- The axiom of symmetry can be replaced by the axiom of *anonymity*:
  Given a game $v$ and a permutation $\pi$ of the players, let $u$ be the game defined as $u(\pi(S)) = v(S)\ \forall\ S\subseteq N$, then $\psi_{\pi(i)}(u) = \psi_i(v)$

- The axiom of dummy player can be replaced by the axiom of *null player*:
  Let $i$ be a null player, i.e. $v(S\cup\{i\}) = v(S), \quad S\subseteq N\setminus\{i\}$ then $\psi_i(v) = 0$

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